Stochastic thermodynamics and information geometry

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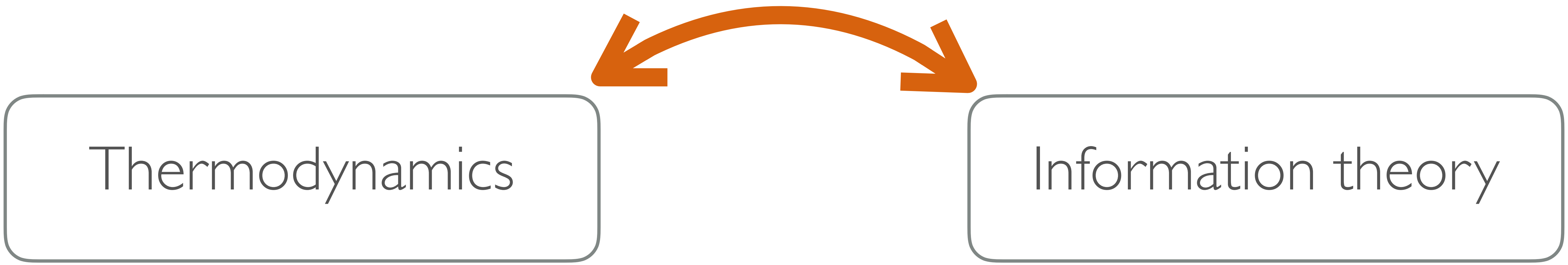
Sosuke Ito, Phys. Rev. Lett. **121**, 030605 (2018), Sosuke Ito and Andreas Dechant, arXiv:1810.06832 (2018).
Sosuke Ito, arXiv:1810.09545 (2018).

Stochastic thermodynamics reveal the relationship b/w thermodynamics and information

Review: Parrondo, J. M., Horowitz, J. M., & Sagawa, T. (2015). Thermodynamics of information. *Nature physics*, *11*(2), 131.

Maxwell's demon



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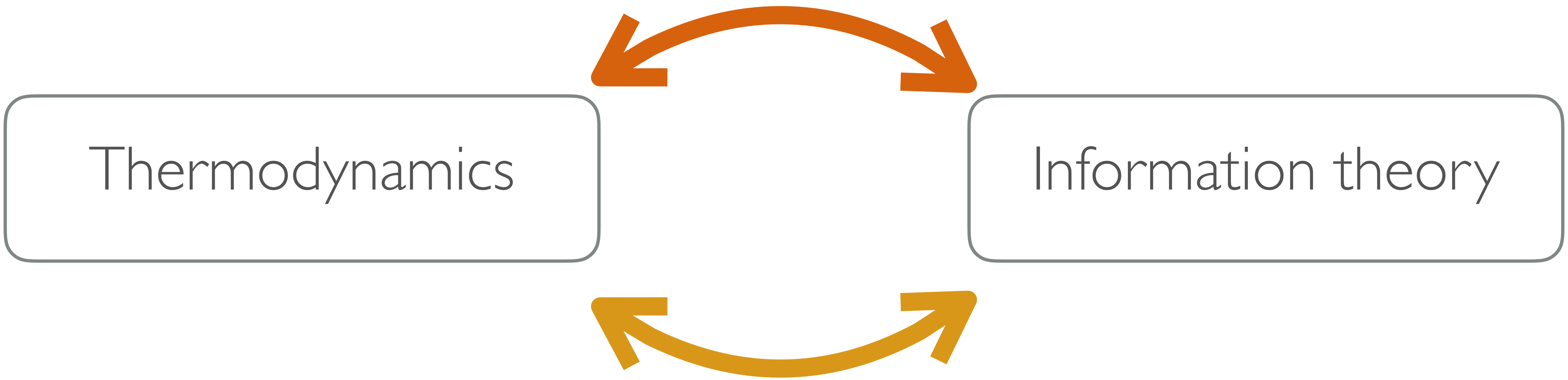

Differential geometry

Ruppeiner geometry(1979)
Thermodynamic length (2007)

Information geometry (1945-)

Ruppeiner, G. (1979). Thermodynamics: A Riemannian geometric model. *Physical Review A*, *20*(4), 1608.

Crooks, G. E. (2007). Measuring thermodynamic length. *Physical Review Letters*, *99*(10), 100602.

Amari, S. I. (2016). *Information geometry and its applications*. Springer Japan.

Differential geometry of thermodynamics

- Ruppeiner geometry (1979)

$$g^{\mathrm{R}}_{ij} = -\partial_{\theta_i}\partial_{\theta_j} S$$

S : Entropy (Thermodynamics)

- Thermodynamic length (2007)

$$g^{\mathrm{F}}_{ij} = -\mathbb{E}[\partial_{\theta_i}\partial_{\theta_j} \ln p_{\mathrm{can}}]$$

p_{can} : Gibbs ensemble

$\mathbb{E}$: Expected value

Differential geometry

$$ds^2 = \sum_{i,j} g_{ij} d\theta_i d\theta_j$$

ds : Line element

g_{ij} : Metric

θ_i : i-th parameter

(near) equilibrium

Ruppeiner, G. (1979). Thermodynamics: A Riemannian geometric model. *Physical Review A*, *20*(4), 1608.

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Information geometry

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- Information geometry (1945, by C. R. Rao) p : probability

$$ds^2 = 2D_{\mathrm{KL}}(p||p+dp) = \sum_{ij} g_{ij} d\theta_i d\theta_j + o(dp^2)$$

KL-divergence: $D_{\mathrm{KL}}(p||q) = \sum_x p_x \ln \frac{p_x}{q_x}$

$$ds^2 = \sum_x \frac{(dp_x)^2}{p_x} = \sum_x (2d\sqrt{p_x})^2$$

- The Fisher metric

$$g_{ij} = \mathbb{E}[\partial_{\theta_i} \ln p \partial_{\theta_j} \ln p] = -\mathbb{E}[\partial_{\theta_i} \partial_{\theta_j} \ln p]$$

→ “non-stationary” dynamics (stochastic thermodynamics)

Sosuke Ito, Phys. Rev. Lett. **121**, 030605 (2018), Sosuke Ito and Andreas Dechant, arXiv:1810.06832 (2018).

Stochastic thermodynamic interpretation of information geometry

- **Master equation**

$$\frac{d}{dt}p_x = \sum_{x',\nu}[W^{(\nu)}_{x'\to x}p_{x'} - W^{(\nu)}_{x\to x'}p_x]$$

$$F^{(\nu)}_{x'\to x} = \ln[W^{(\nu)}_{x'\to x}p_{x'}] - \ln[W^{(\nu)}_{x\to x'}p_x]$$: Force

$$J^{(\nu)}_{x'\to x} = W^{(\nu)}_{x'\to x}p_{x'} - W^{(\nu)}_{x\to x'}p_x$$: Flux

$$\Delta\sigma^{(\nu)}_{\mathrm{bath};x'\to x} = \ln\frac{W^{(\nu)}_{x'\to x}}{W^{(\nu)}_{x\to x'}}$$: Entropy change of the heat bath

$$\left(\frac{ds}{dt}\right)^2 = \left\langle\frac{d\Delta\sigma_{\mathrm{bath}}}{dt}\right\rangle - \left\langle\frac{dF}{dt}\right\rangle$$

$$\langle A\rangle = \sum_{\nu,x,x'|x>x'} J^{(\nu)}_{x'\to x}A^{(\nu)}_{x'\to x}$$

Inequalities

In near equilibrium (F~0),

$$ds^2 = \langle d\Delta\sigma_{\rm bath} \rangle dt$$

Non-negativity

cf.) 2nd law

$$\left(\frac{ds}{dt}\right)^2 = \left\langle \frac{d\Delta\sigma_{\rm bath}}{dt} \right\rangle - \left\langle \frac{dF}{dt} \right\rangle \geq 0$$

$$\langle F \rangle \geq 0$$

Monotonicity

$$\frac{d}{dt}\left[\left(\frac{ds}{dt}\right)^2\right] = \frac{d}{dt}\left[\left\langle \frac{d\Delta\sigma_{\rm bath}}{dt} \right\rangle - \left\langle \frac{dF}{dt} \right\rangle\right] \leq 0 \quad \text{if} \quad \frac{dW^{(\nu)}_{x\to x'}}{dt} = 0$$

for all x, x', ν

Geometric quantities

- **Length**

$$\mathcal{L} = \int_0^{\tau} dt \left(\frac{ds}{dt} \right)$$

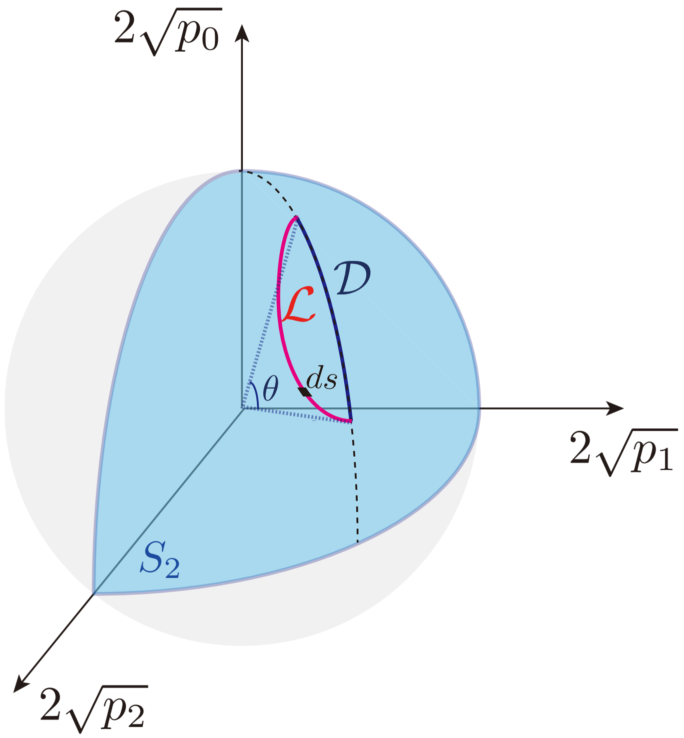
- **Geodesic**

$$\mathcal{L} \geq \mathcal{D}$$

$$\mathcal{D} = 2 \cos^{-1} \left(\sum_i \sqrt{p_i(0)} \sqrt{p_i(\tau)} \right)$$

- **Action (thermodynamic cost)**

$$\mathcal{C} = \frac{1}{2} \int_0^{\tau} dt \left(\frac{ds}{dt} \right)^2$$



Thermodynamic speed limit

- **Speed limit I**

$$2\mathcal{C}\tau \geq \mathcal{L}^2 \geq \mathcal{D}^2$$

Trade-off relationship between C and τ

- **Speed limit II (Monotonicity)**

$$\left|\frac{ds}{dt}\right|_{t=0} \tau \geq \mathcal{D} \quad \text{if} \quad \frac{dW^{(\nu)}_{x\to x'}}{dt} = 0$$

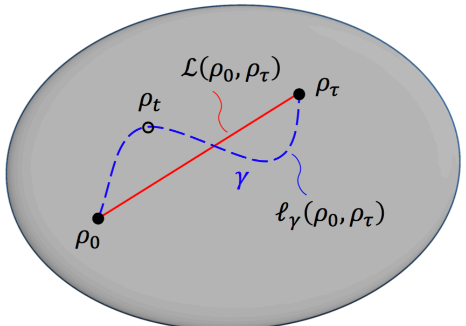
for all x, x', ν

cf.) Quantum speed limit

(Uncertainty relationship between energy and time)

Pires, D. P. Et al. PRX (2016).

$$\tau\sqrt{\mathbb{E}[H^2] - \mathbb{E}[H]^2} \geq \hbar \arccos(\langle\phi(\tau)|\phi(0)\rangle)$$



Thermodynamic uncertainty

- **Cramér-Rao bound**

$$\left(\frac{ds}{dt}\right)^2 (\mathbb{E}[R^2]_t - \mathbb{E}[R]_t^2) \geq \left(\frac{d}{dt}\mathbb{E}[R]_t\right)^2$$

R: Observable of x.

cf.) Themodynamic uncertainty (in s.s.)

e.g.) $\langle \dot{X} \rangle_{\mathrm{st}}^2 \leq D_X \langle \Delta\sigma_{\mathrm{bath}} \rangle_{\mathrm{st}}$

Barato, A. C., & Seifert, U. *PRL (2015).*

Gingrich, T. R., Horowitz, J. M., Perunov, N., & England, J. L. *PRL* (2016).

Maes, C. PRL (2017).

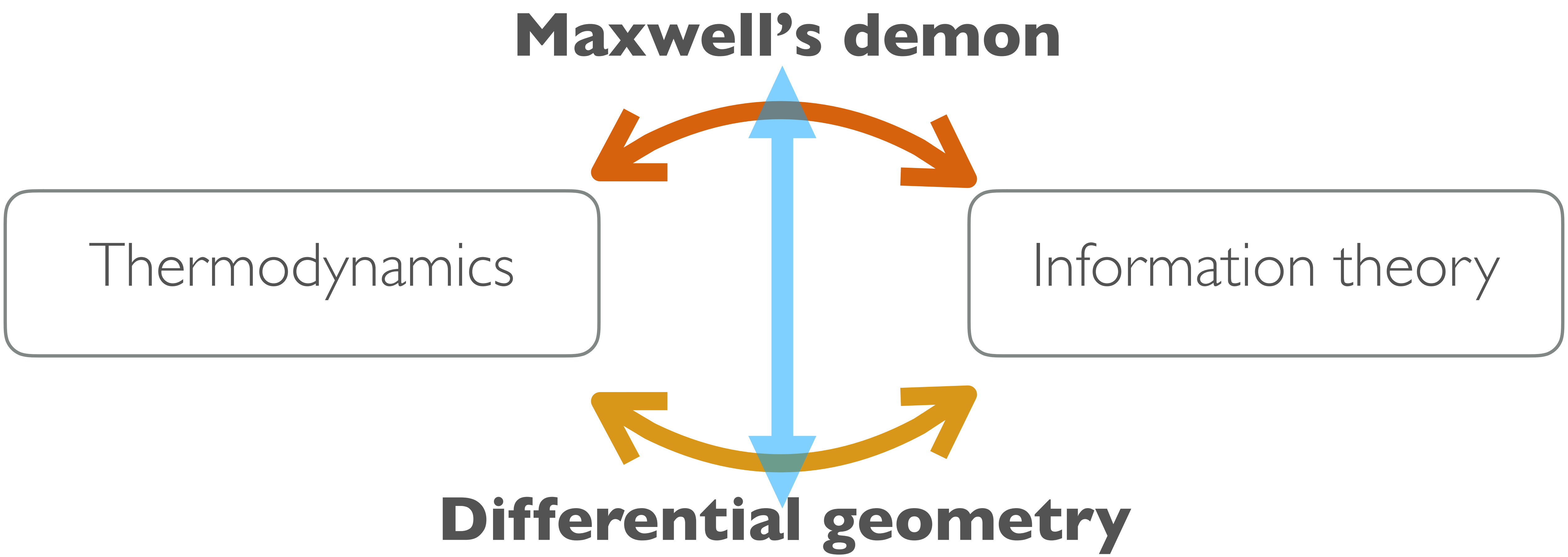
…etc.

- **Speed limit III (F-R ratio)**

$$2\mathcal{C}\tau \geq \frac{(\mathbb{E}[R]_\tau - \mathbb{E}[R]_0)^2}{\max_{0\leq t\leq\tau}[\mathbb{E}[R^2]_t - \mathbb{E}[R]_t^2]}$$

The relationship b/w
Maxwell's demon and information geometry

Sosuke Ito, arXiv:1810.09545 (2018).



The (generalized) second law can be also derived from information geometry.

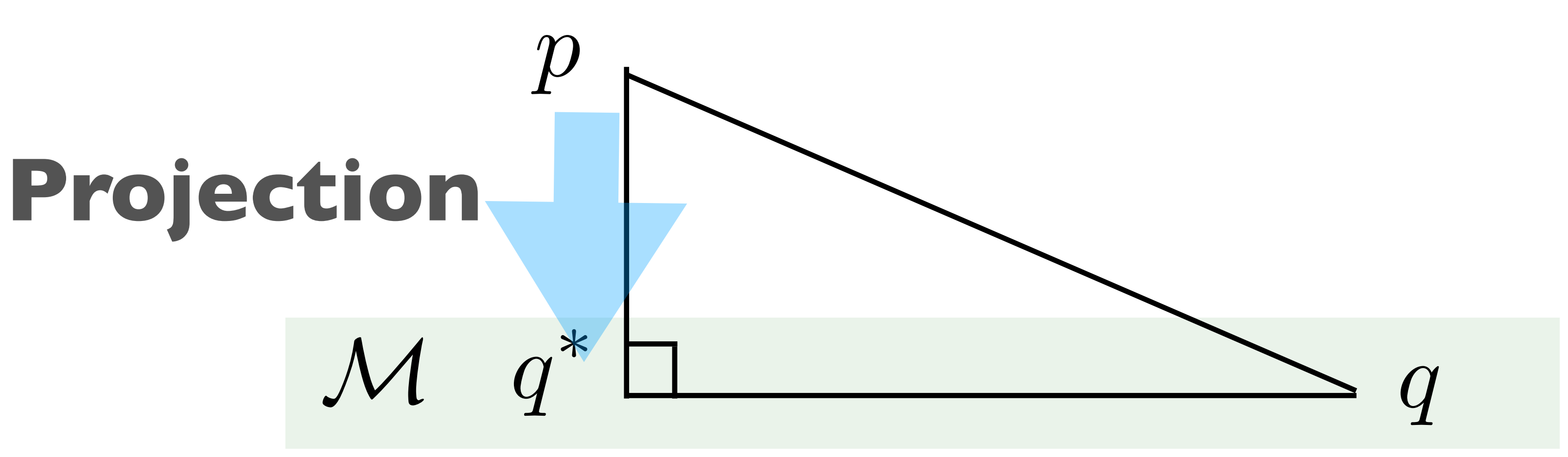
Projection theorem in information geometry

Amari, S. I. (2016). *Information geometry and its applications*. Springer Japan.

Projection theorem

For a smooth and flat manifold M, we have a unique q=q* on M to minimize the KL divergence between p and q.

$$\min_{q \in \mathcal{M}} D_{\mathrm{KL}}(p||q) = D_{\mathrm{KL}}(p||q^*)$$



The second law of (information) thermodynamics and projection

Sosuke Ito, arXiv:1810.09545 (2018).

Projection onto

$\mathcal{M}_R$: the set of reversible dynamics

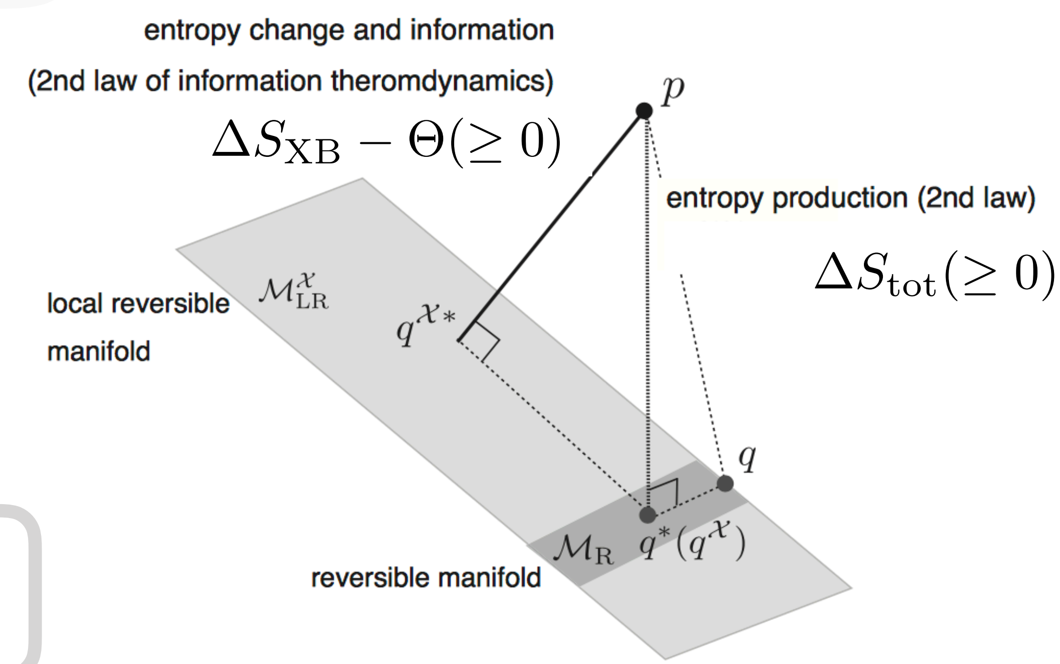
$\mathcal{M}_{LR}^{\mathcal{X}}$: the set of local reversible dynamics in X

2nd law of thermo.

$$\min_{\mathcal{Q}\in M_R} D_{\mathrm{KL}}(\mathcal{P}||\mathcal{Q}) = \Delta S_{\mathrm{tot}} \ (\geq 0)$$

2nd of info. thermo. (Maxwell's demon)

$$\min_{\mathcal{Q}\in M_{LR}^{\mathcal{X}}} D_{\mathrm{KL}}(\mathcal{P}||\mathcal{Q}) = \Delta S_{XB} - \Theta_{\mathrm{info}} \ (\geq 0)$$



ΔS_{tot} : the total entropy production

ΔS_{XB} : the entropy change of bath and the system X

Θ_{info} : the mutual information change (flow)

$\mathcal{P}, \mathcal{Q}$: path probabilities

Hierarchy

$$\mathcal{M}_R \subset \mathcal{M}_{LR}^{\mathcal{X}}$$

$$\Rightarrow \Delta S_{\mathrm{tot}} \geq \Delta S_{XB} - \Theta_{\mathrm{info}}$$

Summary

We consider stochastic thermodynamics of information geometry as a generalization of differential geometry of thermodynamics for non-stationary dynamics.

We obtain new thermodynamic inequalities and thermodynamic speed limits corresponding to the uncertainty relationship (quantum speed limit).

We make a link between stochastic thermodynamics of information geometry and the discussion of Maxwell's demon.

For details:

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