Introduction to stochastic thermodynamics for computer science theorists II: Trade-offs in stochastic thermodynamics

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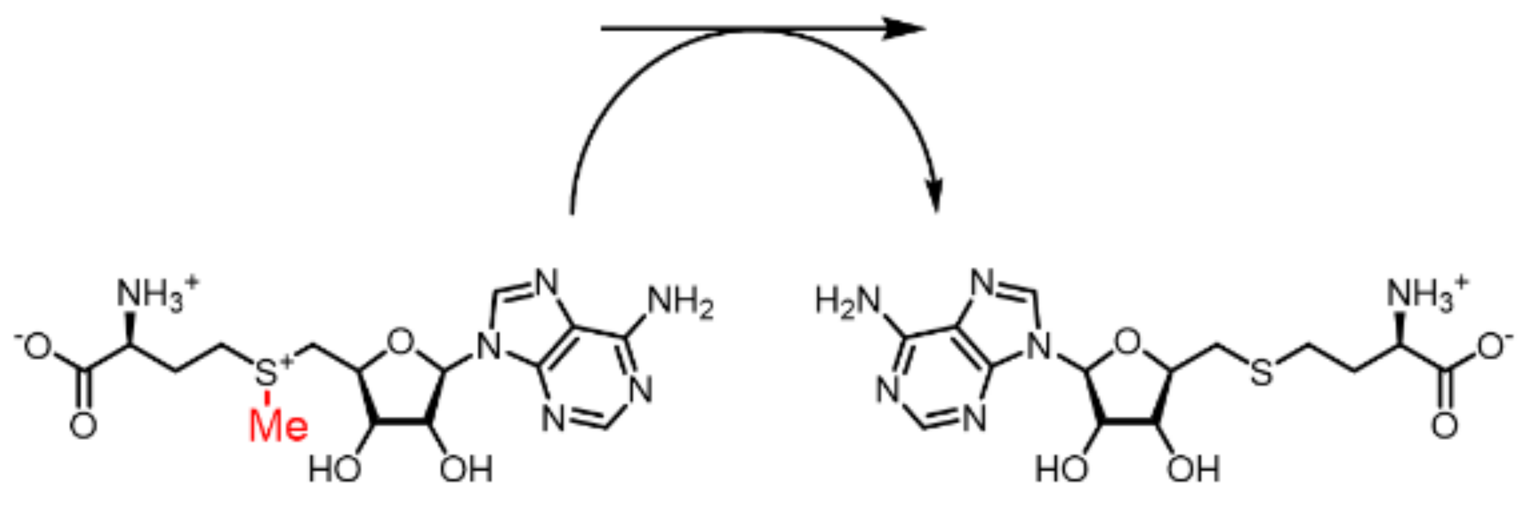
TOPICS

- **Introduction: Trade-offs in stochastic thermodynamics**
- Entropy production rate and its expressions
- Free energy and mismatch costs
- Thermodynamic uncertainty relations
- Wasserstein distance and speed limits
- Other bounds (Kinetic uncertainty relations, correlations …etc.)
- Summary and perspective in computer science

Energetic cost vs performance (speed, accuracy …etc.)

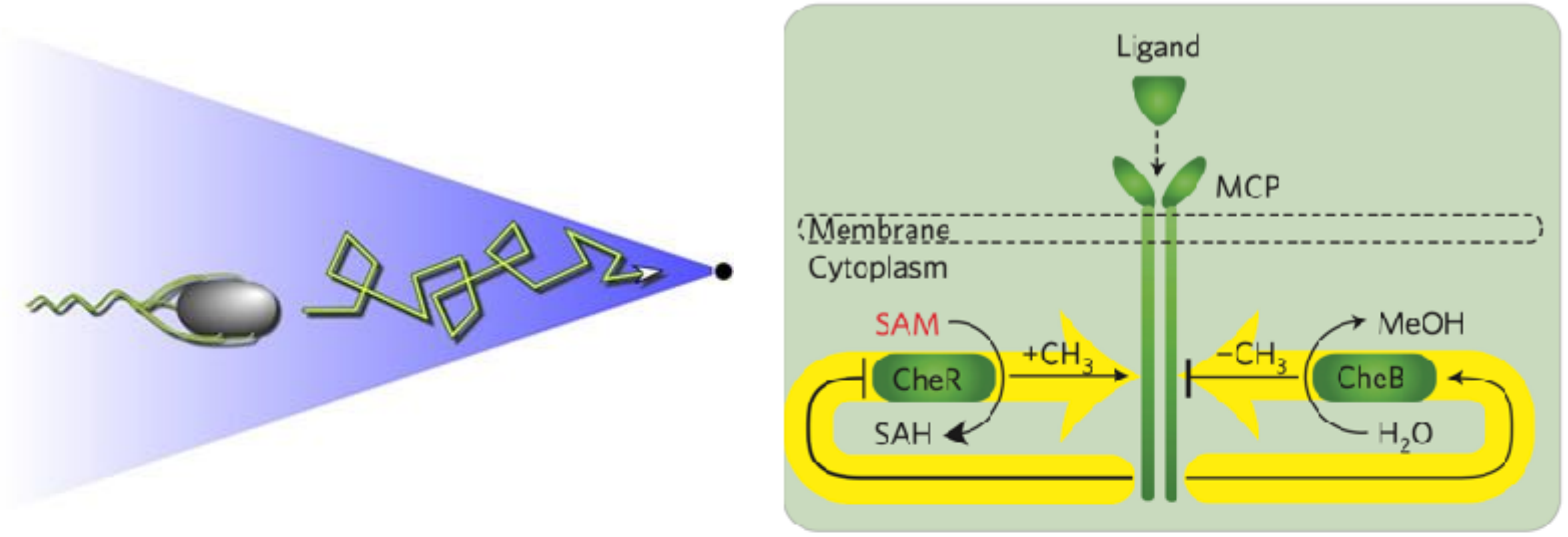
Chemical driving force



S-Adenosylmethionine (SAM) S-Adenosylhomocysteine (SAH)

versus

Sensory performance

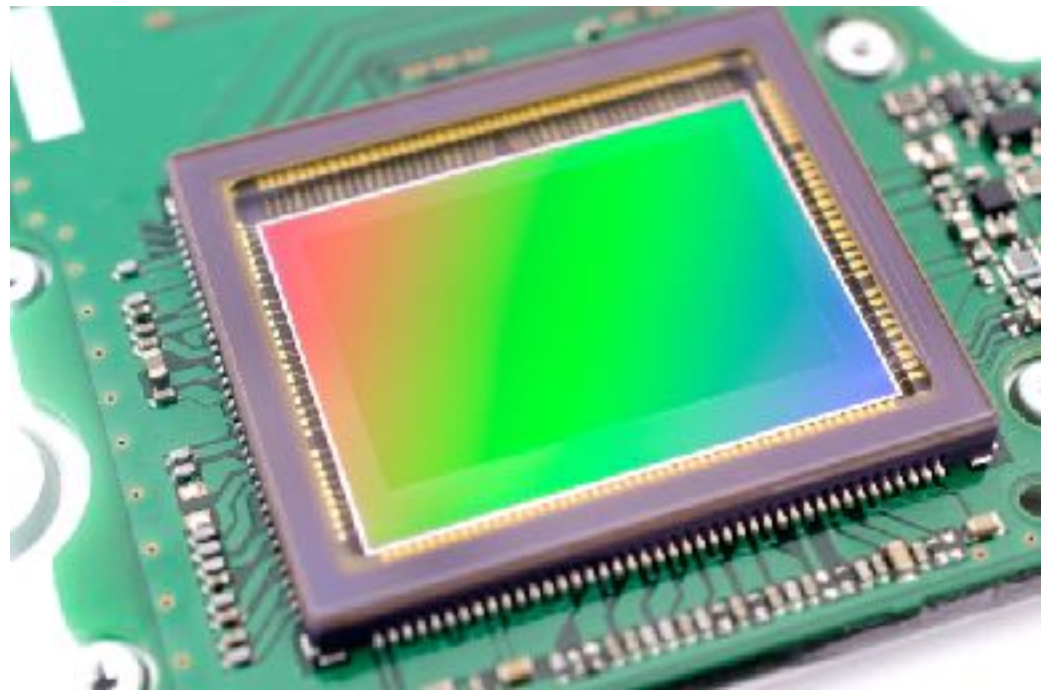


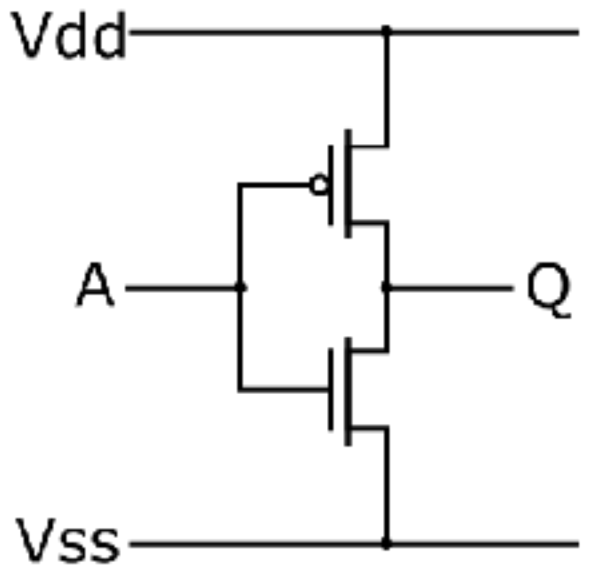
e.g.,) G. Lan, P. Sartori, S. Neumann, V. Sourjik & Y. Tu, *Nature physics*, **8**, 422-428 (2012).

Electric power

versus

Computational performance





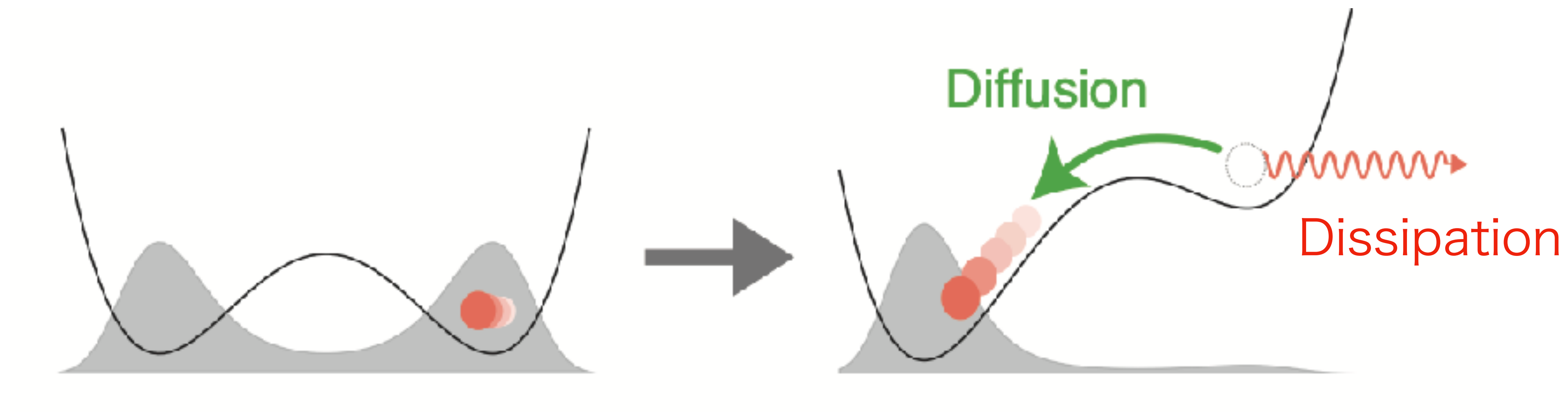
e.g.,) D. H. Wolpert, J. Korbel, C. W. Lynn, F. Tasnim, J. A. Grochow, G. Kardeş, ... & J. Paradiso, *Proceedings of the National Academy of Sciences*, **121**, e2321112121 (2024).
J. C. Delvenne, & L. Van Brandt, *Physical Review Letters*, **134**, 237101 (2025).

Are there any laws regarding energy costs?

Fundamental laws regarding energetic costs can be discussed
in stochastic thermodynamics.

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Nonequilibrium thermodynamics based on stochastic dynamics



e.g.,) Langeivn/Fokker-Planck dynamics

Note:
Some of the results can be generalized to both macroscopic deterministic and quantum systems.
(e.g., Chemical reaction, Fluid dynamics, Reaction-diffusion, Open quantum dynamics …etc.)

Trade-offs in stochastic thermodynamics

Dissipation
(The entropy production rate)

Driving force (Affinity)

Number of jump

…etc.

versus

Fluctuation (accuracy)

Speed of dynamics

Correlation

Response

Oscillation

…etc.

Important example: Thermodynamic uncertainty relations

Dissipation
(The entropy production rate)

Versus

Fluctuation (accuracy)

Dissipation is necessary to reduce fluctuation and perform accurately.

Thermodynamic uncertainty relations

A. C. Barato & U. Seifert, *Physical review letters*, **114**, 158101 (2015).

A lot of variants exists.

Review) J. M. Horowitz & T. R. Gingrich, *Nature Physics*, **16**, 15-20 (2020). …etc.

cf.) Cramér–Rao bound in information theory

The fundamental limits of estimation error.

Fisher information **Versus** Fluctuation (accuracy)

Mathematically speaking, thermodynamic trade-offs are related to several relations in information theory such as the Cramér–Rao bound.

Note:
Several quantum uncertainty relations are also related to several relations in information theory such as the Cramér–Rao bound.

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Entropy production rate for the master equation

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Master equation (Markov jump process)

$$\partial_t p_i(t) = \sum_j W_{ij} p_j(t) = \sum_j [W_{ij} p_j(t) - W_{ji} p_i(t)]$$

Def.) Entropy production rate [without odd degrees of freedom]

$$\sigma = \sum_{i,j} W_{ij} p_j \ln \frac{W_{ij} p_j}{W_{ji} p_i} = \sum_{i,j|i>j} (W_{ij} p_j - W_{ji} p_i) \ln \frac{W_{ij} p_j}{W_{ji} p_i}$$

A measure of irreversibility ($W_{ij} p_j \neq W_{ji} p_i \Rightarrow \sigma > 0$)

Entropy production and Kullback-Leibler divergence

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Entropy production:

$$\Sigma_{[t,t+dt]} := \int_t^{t+dt} dt\sigma(t) \simeq \sum_{i,j} W_{ij}p_j dt \ln \frac{W_{ij}p_j dt}{W_{ji}p_i dt} = \sum_{i,j} \mathbb{P}(i,j) \ln \frac{\mathbb{P}(i,j)}{\mathbb{P}^\dagger(i,j)} =: \underline{D_{\mathrm{KL}}(\mathbb{P}||\mathbb{P}^\dagger)}$$

Kullback-Leibler divergence
(in information theory)

① $D_{\mathrm{KL}}(\mathbb{P}||\mathbb{P}^\dagger) \geq 0$

② $D_{\mathrm{KL}}(\mathbb{P}||\mathbb{P}^\dagger) = 0 \Leftrightarrow \mathbb{P} = \mathbb{P}^\dagger$

Joint probability of j at time t and i at time $t + dt$.

$$\mathbb{P}(i,j) = \begin{cases} W_{ij}p_j(t)dt & (i \neq j) \\ 1 - \sum_{j(\neq i)} W_{ij}p_j(t)dt & (i = j) \end{cases}$$

Joint probability of i at time t and j at time $t + dt$.

$$\mathbb{P}^\dagger(i,j) = \mathbb{P}(j,i)$$

Force-flux structure and entropy production rate

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Thermodynamic driving force: $F_{ij} = \ln \frac{W_{ij}p_j}{W_{ji}p_i}$

Flux: $J_{ij} = W_{ij}p_j - p_iW_{ji}$

Master equation: $\partial_t p_i = \sum_j J_{ij}$

Entropy production rate:

$$\sigma = \sum_{i,j|i>j} (W_{ij}p_j - W_{ji}p_i)\ln \frac{W_{ij}p_j}{W_{ji}p_i} = \sum_{i,j|i>j} J_{ij}F_{ij}$$

Entropy production rate for the Fokker-Planck equation

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Fokker-Planck equation (overdamped)

$$\partial_t P(\boldsymbol{x};t) = -\nabla \cdot (\boldsymbol{\nu}(\boldsymbol{x})P(\boldsymbol{x};t))$$

$$\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - T\nabla \ln P(\boldsymbol{x};t)$$

Def.) Entropy production rate

$$\sigma = \frac{1}{T}\int d\boldsymbol{x} \|\boldsymbol{\nu}(\boldsymbol{x})\|^2 P(\boldsymbol{x})$$

A measure of irreversibility ($\boldsymbol{\nu}(\boldsymbol{x}) \neq \mathbf{0} \Rightarrow \sigma > 0$)

Entropy production and Kullback-Leibler divergence

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Entropy production:

$$\Sigma_{[t,t+dt]} = \int_t^{t+dt} dt \sigma(t) \simeq \int d\boldsymbol{x} d\boldsymbol{y} \mathbb{P}_{\mathrm{FP}}(\boldsymbol{x}, \boldsymbol{y}) \ln \frac{\mathbb{P}_{\mathrm{FP}}(\boldsymbol{x}, \boldsymbol{y})}{\mathbb{P}^{\dagger}_{\mathrm{FP}}(\boldsymbol{x}, \boldsymbol{y})} = D_{\mathrm{KL}}(\mathbb{P}_{\mathrm{FP}} \,||\, \mathbb{P}^{\dagger}_{\mathrm{FP}})$$

Joint probability of $\boldsymbol{y}$ at time t and $\boldsymbol{x}$ at time $t + dt$.

$$\mathbb{P}_{\mathrm{FP}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\sqrt{4\pi T dt}} \exp\left[-\frac{\|\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{y}) dt\|^2}{4T dt} \right] P(\boldsymbol{y}; t)$$

Joint probability of $\boldsymbol{x}$ at time t and $\boldsymbol{y}$ at time $t + dt$.

$$\mathbb{P}^{\dagger}_{\mathrm{FP}}(\boldsymbol{x}, \boldsymbol{y}) = \mathbb{P}_{\mathrm{FP}}(\boldsymbol{y}, \boldsymbol{x})$$

Force-flux structure and entropy production rate

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

Thermodynamic driving force: $\boldsymbol{F}(\boldsymbol{x}) = \frac{\boldsymbol{\nu}(\boldsymbol{x})}{T} = \frac{\boldsymbol{f}(\boldsymbol{x})}{T} - \nabla \ln P(\boldsymbol{x})$

Flux: $\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{\nu}(\boldsymbol{x}) P(\boldsymbol{x})$

Fokker-Planck equation: $\partial_t P(\boldsymbol{x}) = -\nabla \cdot \boldsymbol{J}(\boldsymbol{x})$

Entropy production rate:

$$\sigma = \frac{1}{T} \int d\boldsymbol{x} \|\boldsymbol{\nu}(\boldsymbol{x})\|^2 P(\boldsymbol{x}) = \int d\boldsymbol{x} \boldsymbol{J}(\boldsymbol{x}) \cdot \boldsymbol{F}(\boldsymbol{x})$$

Origin of trade-off relations

Entropy production rate (Dissipation):

$$\sigma = \frac{D_{\mathrm{KL}}(\mathbb{P}\,||\,\mathbb{P}^{\dagger})}{dt} = \sum_{i,j|i>j} J_{ij} F_{ij}$$

Driving force

Relaxation, Response

Oscillation

Information-theoretic quantity

Estimation, Accuracy, Fluctuation

Kinetic quantity

Speed, Distance

Change of the observable

These expressions are mathematical origin of trade-offs.

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Conservative driving force

The thermodynamic driving force is **conservative** if

$\exists p^{\mathrm{eq}}$ s.t. detailed balance condition $W_{ji} p_i^{\mathrm{eq}} = W_{ij} p_j^{\mathrm{eq}}$

Thermodynamic driving force: $F_{ij} = \ln \frac{p_i}{p_i^{\mathrm{eq}}} - \ln \frac{p_j}{p_j^{\mathrm{eq}}}$

The entropy production rate:

$$\sigma = \sum_{i,j|i>j} J_{ij} F_{ij} = -\sum_i \partial_t p_i \ln \frac{p_i}{p_i^{\mathrm{eq}}} = -\partial_t D_{\mathrm{KL}}(p || p^{\mathrm{eq}})$$

(Nonequilibrium) free energy

Review) U. Seifert. *Reports on progress in physics*, **75**, 126001 (2012).

If the thermodynamic driving force is conservative,

$$\sigma = -\partial_t D_{\mathrm{KL}}(p(t)||p^{\mathrm{eq}})$$

$$\Rightarrow \quad \Sigma_{[0,\tau]} = \int_0^\tau dt \sigma(t) = \underline{D_{\mathrm{KL}}(p(\tau)||p^{\mathrm{eq}}) - D_{\mathrm{KL}}(p(0)||p^{\mathrm{eq}})}$$

(Nonequilibrium) free energy difference

$D_{\mathrm{KL}}(p||p^{\mathrm{eq}})$: (Nonequilibrium) free energy of the state p

What happens if the driving force is not conservative? - Several generalizations

Instead of p^{eq}

1. Considering the steady state distribution p^{st}

$$\sigma^{\mathrm{na}} = -\partial_t D_{\mathrm{KL}}(p(t)||p^{\mathrm{st}})\ (\leq \sigma)$$

Non-adiabatic (Hatano-Sasa excess) entropy production rate

T. Hatano, & S. I. Sasa, *Physical review letters*, **86**, 3463 (2001).
M. Esposito & C. Van den Broeck. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, **82**, 011143 (2010).

2. Considering the imaginary conservative force that mimics dynamics

$$\sigma^{\mathrm{ex}} = -\partial_t D_{\mathrm{KL}}(p(t)||p^{\mathrm{peq}}(s))\Big|_{s=t}\ (\leq \sigma)$$

$$F_{ij}^{\mathrm{conservative}} = \ln \frac{p_i}{p_i^{\mathrm{peq}}} - \ln \frac{p_j}{p_j^{\mathrm{peq}}}$$

Onsager-geometric (Maes-Netocny) excess entropy production rate

C. Maes & K. A. Netočný, *Journal of Statistical Physics*, **154**, 188-203 (2014).
A. Dechant, S. I. Sasa & S. Ito, *Physical Review E*, **106**, 024125 (2022).
K. Yoshimura, A. Kolchinsky, A. Dechant, & S. Ito, *Physical Review Research*, **5**, 013017 (2023).

3. Considering the distribution q_0 which gives the minimum entropy production

Mismatch cost

A. Kolchinsky, & D. H. Wolpert, *Journal of Statistical Mechanics: Theory and Experiment*, **2017**, 083202 (2017).
D. H. Wolpert & A. Kolchinsky, *New Journal of Physics*, **22**, 063047 (2020).
A. Kolchinsky & D. H. Wolpert, *Physical Review E*, **104**, 054107 (2021).

Mismatch cost

A. Kolchinsky, & D. H. Wolpert, *Journal of Statistical Mechanics: Theory and Experiment*, **2017**, 083202 (2017).
D. H. Wolpert & A. Kolchinsky, *New Journal of Physics*, **22**, 063047 (2020).
A. Kolchinsky & D. H. Wolpert, *Physical Review E*, **104**, 054107 (2021).

$$\Sigma(p) = \sum_i \Phi(p)_i \ln \Phi(p)_i - \sum_i p_i \ln p_i - \sum_i p_i Q_i$$

: The entropy production (General expression)

$$\Phi(p)_i = \sum_j T_{ij} p_j$$

T_{ij}: Transition probability

Q_i: Entropy change in the heat bath

Probability distribution which gives the minimum entropy production:

$$q = \mathrm{argmin}_p \Sigma(p)$$

$$\Sigma(p) = \Sigma(q) + D_{\mathrm{KL}}(p\|q) - D_{\mathrm{KL}}(\Phi(p)\|\Phi(q)) \geq \underline{D_{\mathrm{KL}}(p\|q) - D_{\mathrm{KL}}(\Phi(p)\|\Phi(q))}$$

Mismatch cost

A merit of mismatch cost (as far as I thought.)

A. Kolchinsky, & D. H. Wolpert, *Journal of Statistical Mechanics: Theory and Experiment*, **2017**, 083202 (2017).
D. H. Wolpert & A. Kolchinsky, *New Journal of Physics*, **22**, 063047 (2020).
A. Kolchinsky & D. H. Wolpert, *Physical Review E*, **104**, 054107 (2021).

$$\Sigma(p) = \sum_i \Phi(p)_i \ln \Phi(p)_i - \sum_i p_i \ln p_i - \sum_i p_i Q_i$$

$$\Phi(p)_i = \sum_j T_{ij} p_j \qquad q = \mathrm{argmin}_p \Sigma(p)$$

We do not need to consider the detail of dynamics.

T_{ij} and Q_i can be arbitrary selected.

$\Sigma(p)$ can not be the entropy production for general T_{ij} and Q_i.

$$D_{\mathrm{KL}}(p \| q) - D_{\mathrm{KL}}(\Phi(p) \| \Phi(q)) (\geq 0)$$

is a genuine information-theoretic quantity, and a lower bound on $\Sigma(p)$.

An application to computations (without dynamics)

G. Manzano, G. Kardeş, É. Roldán, & D. H. Wolpert, *Physical Review X*, **14**, 021026 (2024).

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Original idea of the thermodynamic uncertainty relations (TURs)

A. C. Barato, & U. Seifert, *Physical review letters*, ***114***, 158101 (2015).

Change of the state (flux)

$$\sigma = \sum_{i,j|i>j} (W_{ij}p_j - W_{ji}p_i)\ln \frac{W_{ij}p_j}{W_{ji}p_i} \geq \sum_{i,j|i>j} \frac{2(W_{ij}p_j - W_{ji}p_i)^2}{(W_{ij}p_j + W_{ji}p_i)}$$

Fluctuation of the state (dynamical activity)

∵) Log-mean vs arithmetic mean: $\frac{a+b}{2} \geq \frac{a-b}{\ln a - \ln b}$

"Dissipation is necessary to reduce fluctuation and perform accurately."

The thermodynamic uncertainty relations (TURs) for observable

TURs for the general current in the steady state

$$\Sigma_{[0,T]} = \int_0^\tau dt\sigma \geq 2\frac{\langle \tilde{J}_r \rangle}{\mathrm{Var}[\tilde{J}_r]}$$

Expected integrated current for observable r

Variance of integrated current r

Conjecture: P. Pietzonka, F. Ritort & U. Seifert, *Physical Review E*, **96**, 012101 (2017).
Proof: J. M. Horowitz & T. R. Gingrich, *Physical Review E*, **96**, 020103 (2017).

TURs for the general current (not in the steady-state)

$$\sigma \geq \sum_{i,j|i>j} \frac{2[r_{ij}(W_{ij}p_j - W_{ji}p_i)]^2}{r_{ij}^2(W_{ij}p_j + W_{ji}p_i)} := \frac{2\langle J_r \rangle^2}{dt\mathrm{Var}[J_r]}$$

Expected current for observable r_{ij}

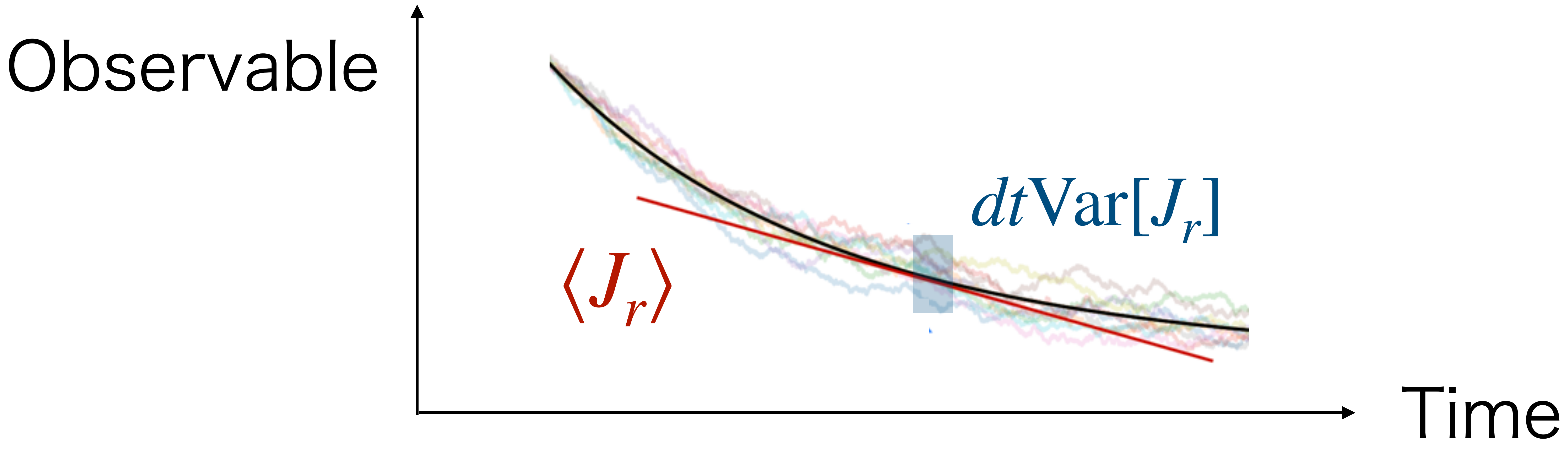
Variance of current for observable r_{ij}

S. Otsubo, S. Ito, A. Dechant, & T. Sagawa, *Physical Review E*, **101**, 062106 (2020).

Physical meaning of the TURs

$$\sigma \geq \frac{2\langle J_r \rangle^2}{dt\mathrm{Var}[J_r]} \quad \Rightarrow \quad \sigma \frac{dt\mathrm{Var}[J_r]}{\langle J_r \rangle^2} \geq 2$$

Dissipation σ is necessary to reduce fluctuation $\mathrm{Var}[J_r]$ and perform accurately ($\langle J_r \rangle \neq 0$).



Information-theoretic viewpoints: Derivation of the TURs from the Chapman-Robbins bounds

An expression in M. Aguilera, S. Ito, & A. Kolchinsky, *arXiv:2505.10444*.

$$\sigma dt = \frac{1}{2}\sum_{i,j}[\mathbb{P}(i,j)-\mathbb{P}^\dagger(i,j)]\ln\frac{\mathbb{P}(i,j)}{\mathbb{P}^\dagger(i,j)} \geq \sum_{i,j}\frac{[\mathbb{P}(i,j)-\mathbb{P}^\dagger(i,j)]^2}{\mathbb{P}(i,j)+\mathbb{P}^\dagger(i,j)}$$

$$= 2\sum_{i,j}\underbrace{\frac{[\mathbb{P}(i,j)-(\mathbb{P}(i,j)+\mathbb{P}^\dagger(i,j))/2]^2}{(\mathbb{P}(i,j)+\mathbb{P}^\dagger(i,j))/2}}_{\chi^2\text{-divergence}} \geq \frac{1}{2}(\langle \boldsymbol{r}\rangle_{\mathbb{P}}-\langle \boldsymbol{r}\rangle_{\mathbb{P}^\dagger})^{\mathsf{T}}[\mathrm{Cov}_{\frac{\mathbb{P}+\mathbb{P}^\dagger}{2}}[\boldsymbol{r}]]^{-1}(\langle \boldsymbol{r}\rangle_{\mathbb{P}}-\langle \boldsymbol{r}\rangle_{\mathbb{P}^\dagger})$$

$\because$ Chapman-Robbins bound for any sets of observable $\boldsymbol{r}_{ij}$

If $\mathbb{P}=\mathbb{P}^\dagger + d\mathbb{P}$, $\sigma dt \simeq \sum_{i,j}\frac{[d\mathbb{P}(i,j)]^2}{\mathbb{P}(i,j)}$ is the Fisher information.

The TUR can also be regarded as the Cramér-Rao bound.

A. Dechant, *Journal of Physics A: Mathematical and Theoretical*, **52**, 035001 (2018).
Y. Hasegawa, & T. Van Vu, Physical Review E, **99**, 062126 (2019).

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p-Wasserstein distance

Textbook) C. Villani, *Optimal transport: old and new*. Berlin: springer (2009).

Distance between two probability distributions $P(\boldsymbol{x}), Q(\boldsymbol{y})$ $[p \geq 1]$

Joint probability $\pi(\boldsymbol{x}, \boldsymbol{y}) \geq 0$

$$\mathscr{W}_p(P, Q) = \left(\inf_{\pi} \int d\boldsymbol{x} \int d\boldsymbol{y} \|\boldsymbol{x} - \boldsymbol{y}\|^p \pi(\boldsymbol{x}, \boldsymbol{y}) \right)^{1/p}$$

s. t.

$$\int d\boldsymbol{y} \pi(\boldsymbol{x}, \boldsymbol{y}) = P(\boldsymbol{x}) \qquad \int d\boldsymbol{x} \pi(\boldsymbol{x}, \boldsymbol{y}) = Q(\boldsymbol{x})$$

A metric:

① $\mathscr{W}_p(P, Q) \geq 0$

② $\mathscr{W}_p(P, Q) = 0 \Leftrightarrow P = Q$

③ $\mathscr{W}_p(P, Q) = \mathscr{W}_p(Q, P)$

④ $\mathscr{W}_p(P, Q) \leq \mathscr{W}_p(P, P') + \mathscr{W}_p(P', Q)$

Hölder-type inequality:

$$p \geq p' \Rightarrow \mathscr{W}_p(P, Q) \geq \mathscr{W}_{p'}(P, Q)$$

1-Wasserstein distance and 2-Wasserstein distance

Textbook) C. Villani, *Optimal transport: old and new*. Berlin: springer (2009).

Duality theorem (Kantorovich-Rubinstein duality)

$$\mathscr{W}_1(P,Q) = \sup_{\psi}\left[\int d\boldsymbol{x} P(\boldsymbol{x})\psi(\boldsymbol{x}) - \int d\boldsymbol{y} Q(\boldsymbol{y})\psi(\boldsymbol{y})\right] \quad \text{s. t.} \quad \|\nabla\psi(\boldsymbol{x})\| \leq 1$$

Fluid-dynamic formulation (Benamou-Brenier formula)

J. D. Benamou & Y. Brenier, *Numerische Mathematik*, **84**, 375-393 (2000)

$$\mathscr{W}_2(P,Q) = \sqrt{\inf_{\boldsymbol{u}_t, Q_t} \tau \int_0^{\tau} dt \int d\boldsymbol{x} \|\boldsymbol{u}_t(\boldsymbol{x})\|^2 Q_t(\boldsymbol{x})} \quad \text{s. t.} \quad \begin{array}{l} \partial_t Q_t(\boldsymbol{x}) = -\nabla\cdot(\boldsymbol{u}_t(\boldsymbol{x})Q_t(\boldsymbol{x})) \\ Q_0(\boldsymbol{x}) = P(\boldsymbol{x}) \quad Q_\tau(\boldsymbol{x}) = Q(\boldsymbol{x}) \end{array}$$

Speed limits for Langevin systems

Distance between initial and final states

$$\Sigma_{[0,\tau]} = \int_0^\tau dt \frac{1}{T} \int d\boldsymbol{x} \|\boldsymbol{\nu}(\boldsymbol{x})\|^2 P(\boldsymbol{x}) \geq \frac{[\mathscr{W}_2(P_0, P_\tau)]^2}{T\tau}$$

Transition time

↑ Benamou-Brenier formula

$$\sigma_t \geq \frac{1}{T} \left[\lim_{\Delta t \to +0} \frac{\mathscr{W}_2(P_t, P_{t+\Delta t})}{\Delta t} \right]^2$$

Speed

"The faster the speed,
the more dissipation is required."

E. Aurell, K. Gawędzki, C. Mejía-Monasterio, R. Mohayaee, & P. Muratore-Ginanneschi, *Journal of statistical physics*, **147**, 487-505 (2012).
M. Nakazato, & S. Ito, *Physical Review Research*, **3**, 043093 (2021).

An analogy of quantum speed limits
[e.g., Uncertainty relation between energy and time]

L. Mandelstam and I. Tamm, *J. Phys.* **9**, 249 (1945).

Speed limits for observables

A. Dechant, S. I. Sasa, & S. Ito, *Physical Review Research*, **4**, L012034 (2022).

$\psi(\boldsymbol{x})$:Observable

$$\sigma_t \geq \frac{1}{T}\left[\lim_{\Delta t \to +0} \frac{\mathcal{W}_2(P_t, P_{t+\Delta t})}{\Delta t}\right]^2 \geq \frac{|d_t \langle \psi \rangle|^2}{T \int d\boldsymbol{x} \|\nabla \psi(\boldsymbol{x})\|^2 P(\boldsymbol{x})}$$

"The faster the speed (of any observable), the more dissipation is required."

- TURs for $\nabla \psi(\boldsymbol{x})$ A. Dechant, & S. I. Sasa, *Journal of Statistical Mechanics: Theory and Experiment*, **2018**, 063209 (2018).

- $\mathcal{W}_2(P_{t+dt}, P_t)/dt \geq \mathcal{W}_1(P_{t+dt}, P_t)/dt = \sup_{\psi} d_t \langle \psi \rangle \ \mathrm{s.t.}\ \|\nabla \psi(\boldsymbol{x})\| \leq 1$

R. Nagayama, K. Yoshimura, A. Kolchinsky & S. Ito, *arXiv:2311.16569,* to appear in Phys. Rev. Res.

- $\psi(\boldsymbol{x}) = \mathrm{sgn}(d_t p(\boldsymbol{x})) \Longrightarrow |d_t \langle \psi \rangle|$ is the speed of the total variation.

Speed limits based on the total variation: N. Shiraishi, K. Funo, & K. Saito, *Physical review letters*, **121**, 070601 (2018).

Speed limits for Markov jump processes

- Based on 1-Wasserstein distance

A. Dechant, *Journal of Physics A: Mathematical and Theoretical*, **55**, 094001 (2022).
T. Van Vu & K. Saito, *Physical Review X* **13**, 011013 (2023).

- Based on 2-Wasserstein distance
(Tighter than the bound by 1-Wasserstein distance)

K. Yoshimura, A. Kolchinsky, A. Dechant, & S. Ito, *Physical Review Research*, **5**, 013017 (2023).

1-Wasserstein distance (Kantrovich-Rubinstein duality)

$\mathscr{W}_1(p,q) = \inf_J \sum_e |J_e|$ s.t. $p_i - q_i = \sum_e \nabla_{ie} J_e$ $\nabla_{ie}^\top$: Incidence matrix

2-Wasserstein distance (A generalization of Benamou-Brenier formula)

$\mathscr{W}_2(p(0),(\tau)) = \sqrt{\inf_{J,p} \tau \int_0^\tau dt \sum_e J_e l_e^{-1}(p) J_e}$ s.t. $\partial_t p_i = \sum_e \nabla_{ie} J_e$ $l_e = J_e/F_e$: Onsager coefficient

Expressions in R. Nagayama, K. Yoshimura, & S. Ito, *Physical Review Research*, **7**, 013307 (2025).

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Kinetic uncertainty relations (not thermodynamic)

"The number of the jump also gives the bound."

$\langle N_\tau \rangle$: The total number of jump up to (the first passage) time τ

A_τ :Any observable of path from $t = 0$ to $t = \tau$

J. P. Garrahan, *Physical Review E*, **95**, 032134 (2017).
I. Di Terlizzi & M. Baiesi, *Journal of Physics A: Mathematical and Theoretical*, **52**, 02LT03 (2018).

$$W_{ij} \to (1+\theta) W_{ij} \qquad \langle N_\tau \rangle \geq \frac{\left[\partial_\theta \langle A_\tau \rangle_\theta \Big|_{\theta=0}\right]^2}{\mathrm{Var}_{\theta=0}[A_\tau]}$$

The derivation is based on the Cramér-Rao bound.

K. Hiura & S. I. Sasa, *Physical Review E*, **103**, L050103 (2021).

The kinetic uncertainty relation can be tighter than the TURs in a far-from-equilibrium state.

Speed limits based on the Fisher information of time

"The speed in information geometry also gives the bound."

$$\frac{ds^2}{dt^2} = \int d\boldsymbol{x} P(\boldsymbol{x})[d_t \ln P(\boldsymbol{x})]^2$$

: Fisher information of time/
Square of the speed in information geometry

Speed limits

$$\int_0^\tau dt \frac{ds^2}{dt^2} \geq \left(\int_0^\tau dt \frac{ds}{dt} \right)^2 \geq 2 \arccos \left(\int d\boldsymbol{x} \sqrt{P_0(\boldsymbol{x}) P_\tau((\boldsymbol{x})} \right)$$

G. E. Crooks, *Physical Review Letters*, **99**, 100602 (2007).
S. Ito, *Physical review letters*, **121**, 030605 (2018).

Speed limits for observable

$$\frac{ds^2}{dt^2} \geq \frac{|d_t \langle r \rangle|^2}{\mathrm{Var}[r]}$$

S. Ito, & A. Dechant, *Physical Review X*, **10**, 021056 (2020).
S. B. Nicholson, L. P. García-Pintos, A. del Campo & J. R. Green, *Nature Physics*, **16**, 1211-1215 (2020).

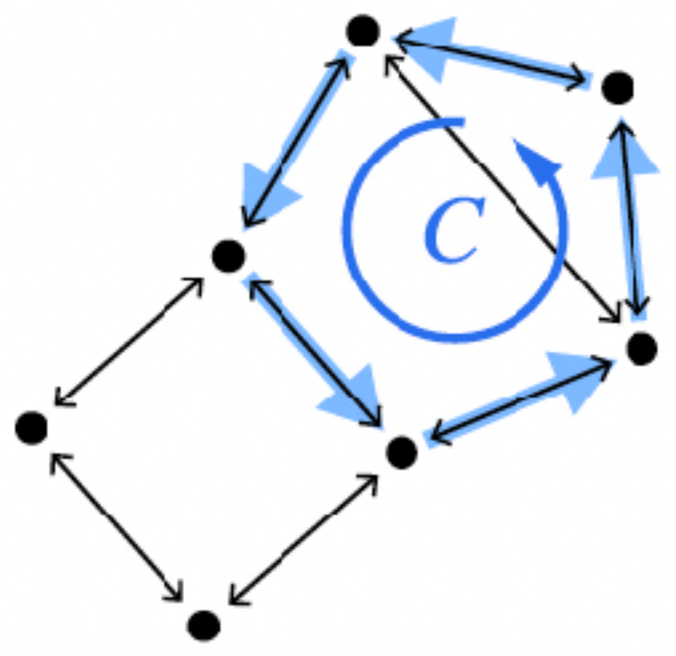
Similar to the results of the Wasserstein-type speed limits

Thermodynamic cycle force and response

J. A. Owen, T. R. Gingrich & J. M. Horowitz, *Physical Review X*, **10**, 011066 (2020).

Cycle force (Non-conservativeness)

$$F_c^{\mathrm{cyc}} = \ln \frac{W_{i_1 i_2} W_{i_2 i_3} \cdots W_{i_{n_c} i_1}}{W_{i_2 i_1} W_{i_3 i_2} \cdots W_{i_1 i_{n_c}}} \qquad c = (i_1 \to i_2 \to i_3 \to \cdots \to i_{n_c} \to i_1)$$



Parameter of the transition rate B_{ij}

$$W_{ij} = \exp(-[B_{ij} - F_{ij}^{\mathrm{th}}/2]) \qquad (B_{ij} = B_{ji})$$

Response of the steady state distribution

$$\left| \frac{\partial p_k^{\mathrm{st}}}{\partial B_{ij}} \right| \le p_k^{\mathrm{st}} \left(1 - p_k^{\mathrm{st}}\right) \tanh \left[\frac{\max_c |F_c^{\mathrm{cyc}}|}{4} \right]$$

"The cycle force is needed for the static response."

Thermodynamic cycle force and eigenvalues (oscillation)

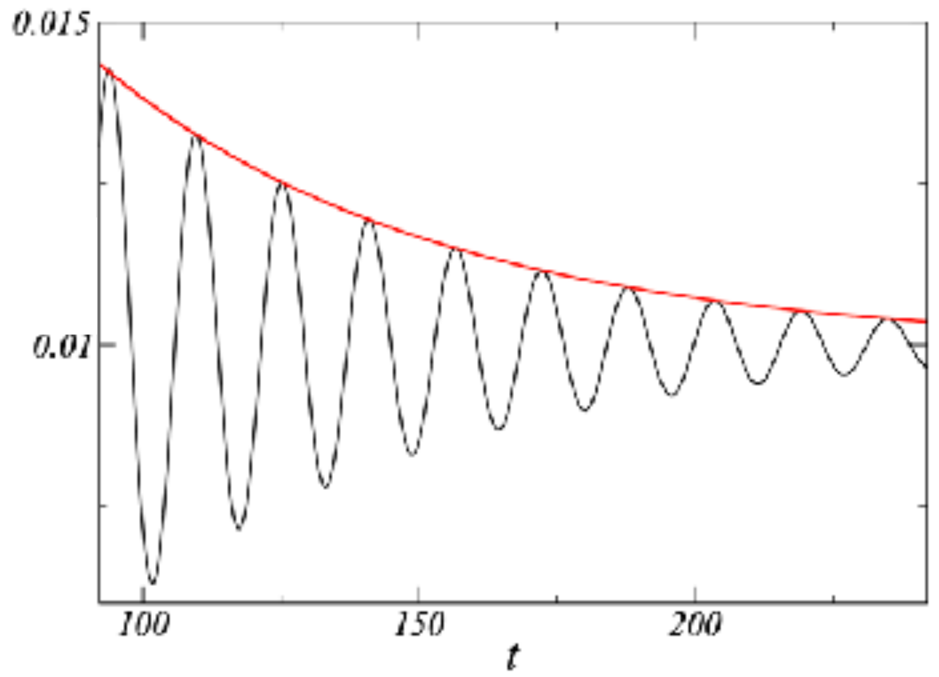
Conjecture for 2nd eigenvalue: A. C. Barato and U. Seifert, Physical Review E, **95**, 062409 (2017).

Proof for any eigenvalues: N Ohga, S. Ito, A. Kolchinsky, Physical Review Letters, **131**, 077101 (2023).

$$\frac{|\mathrm{Im}\lambda|}{2\pi|\mathrm{Re}\lambda|}$$:Ratio between imaginary and real eigenvalues of W_{ij}

(Number of coherent oscillation for 2nd eigenvalue)

$$\frac{\text{(Decay time)}}{\text{(Period of oscillations)}} = \frac{|\mathrm{Im}\lambda|}{2\pi|\mathrm{Re}\lambda|}$$



Thermodynamic bound

$$\frac{|\mathrm{Im}\lambda|}{2\pi|\mathrm{Re}\lambda|} \leq \max_c \frac{\tanh[F_c^{\mathrm{cyc}}/(2n_c)]}{2\pi\tan(\pi/n_c)}$$

n_c : Number of the state in a cycle

"The cycle force is needed for the oscillatory behavior in the relaxation."

Thermodynamic cycle force and asymmetry of cross-correlations

N Ohga, S. Ito, A. Kolchinsky, Physical Review Letters, **131**, 077101 (2023).

Asymmetry of cross-correlations

$$\chi_{ba} = \lim_{\tau \to 0} \frac{C^{\tau}_{ba} - C^{\tau}_{ab}}{2\sqrt{(C^{\tau}_{aa} - C^{0}_{aa})(C^{\tau}_{bb} - C^{0}_{bb})}}$$

Decay of auto-correlations

Cross-correlations: $C^{\tau}_{ba} = \langle b(t+\tau)a(t) \rangle_{\mathrm{st}}$

Thermodynamic bound

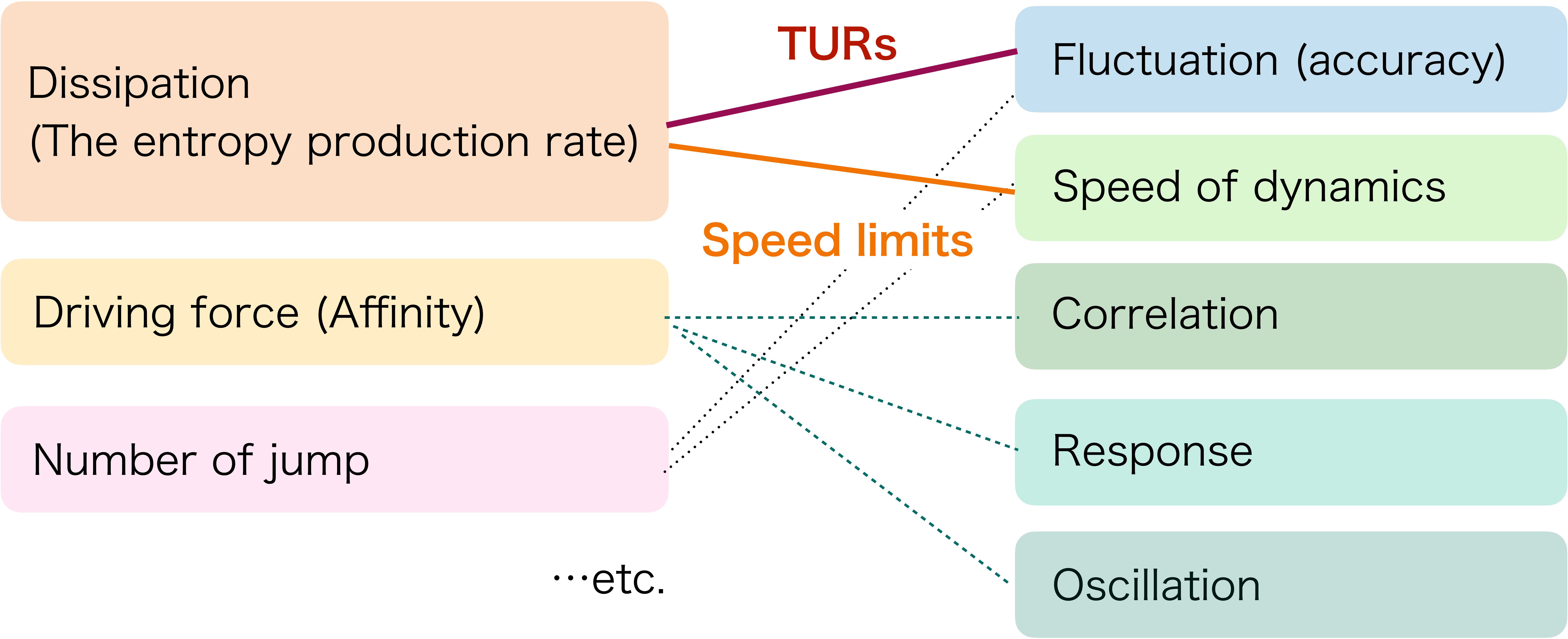
$$|\chi_{ba}| \leq \max_{c} \frac{\tanh[F^{\mathrm{cyc}}_{c}/(2n_c)]}{2\pi \tan(\pi/n_c)}$$

"The cycle force is needed for the asymmetric information transmission."

TOPICS

- Introduction: Trade-offs in stochastic thermodynamics
- Entropy production rate and its expressions
- Free energy and mismatch costs
- Thermodynamic uncertainty relations
- Wasserstein distance and speed limits
- Other bounds (Kinetic uncertainty relations, correlations …etc.)
- **Summary and perspective in computer science**

Summary: Trade-offs in stochastic thermodynamics

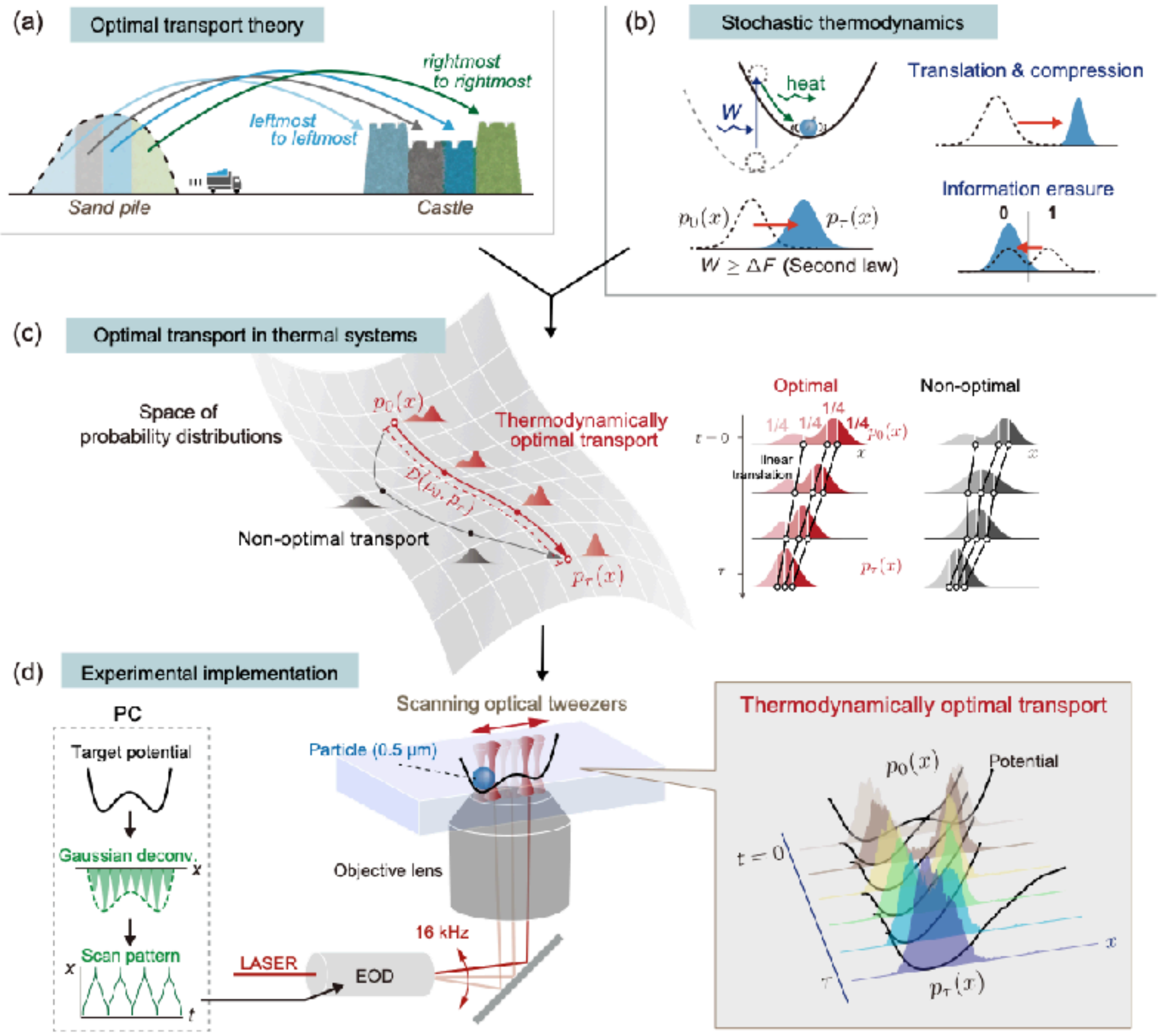


Q: Is it useful in the field of computer science?

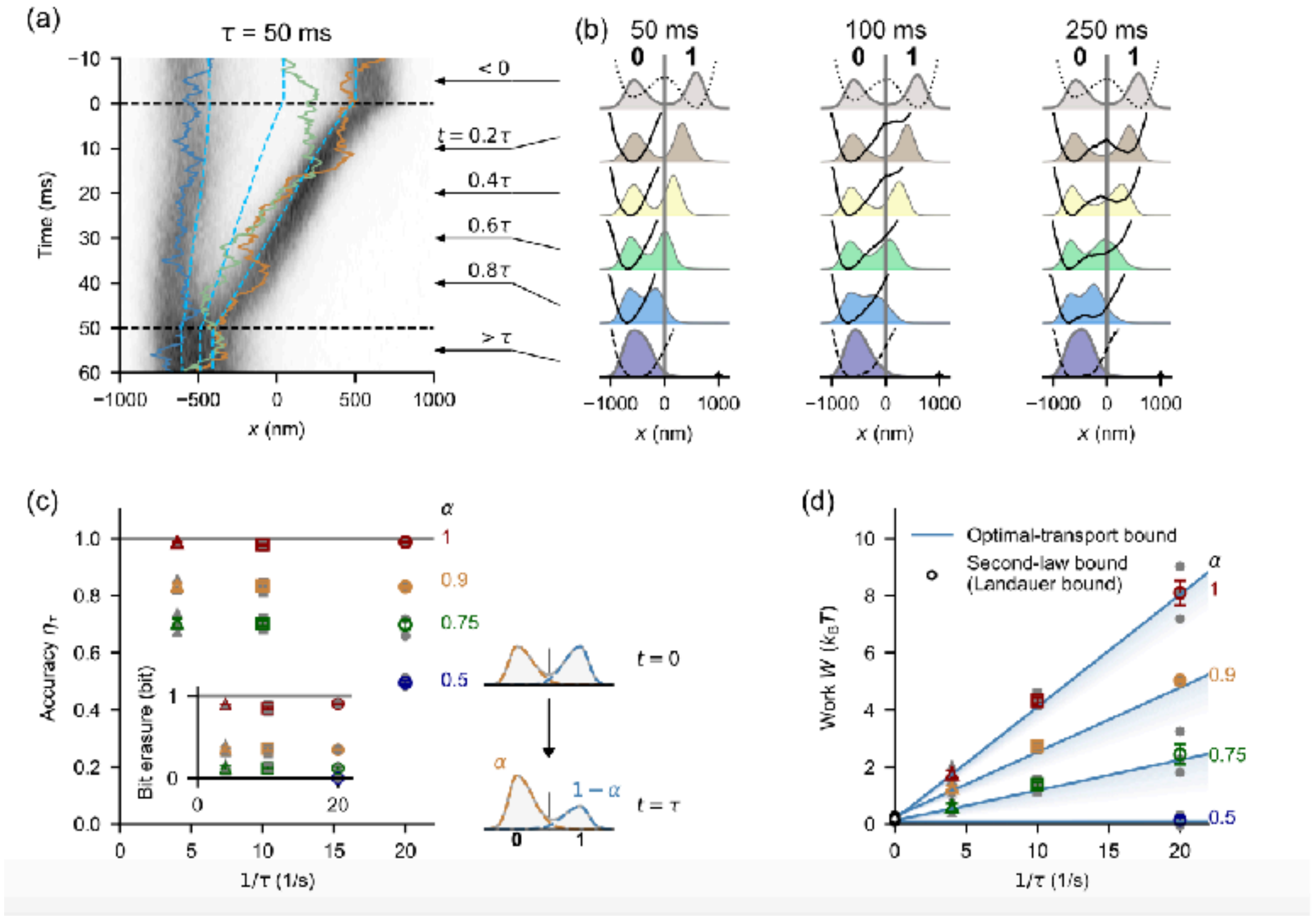
…etc.

Our works: Possible approach for computer science I

S. Oikawa, Y. Nakayama, S. Ito, T. Sagawa, & S. Toyabe, *arXiv:2503.01200.*



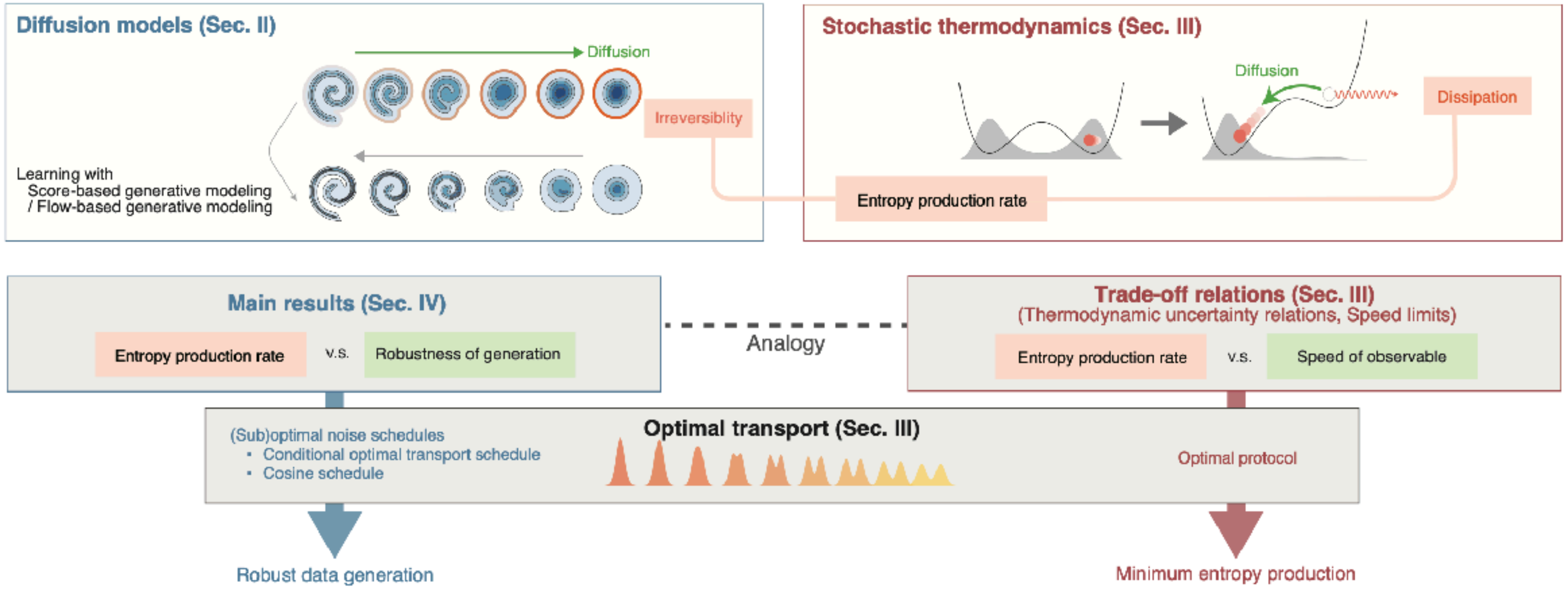
Experiment on speed limits for an energy-efficient information processing device

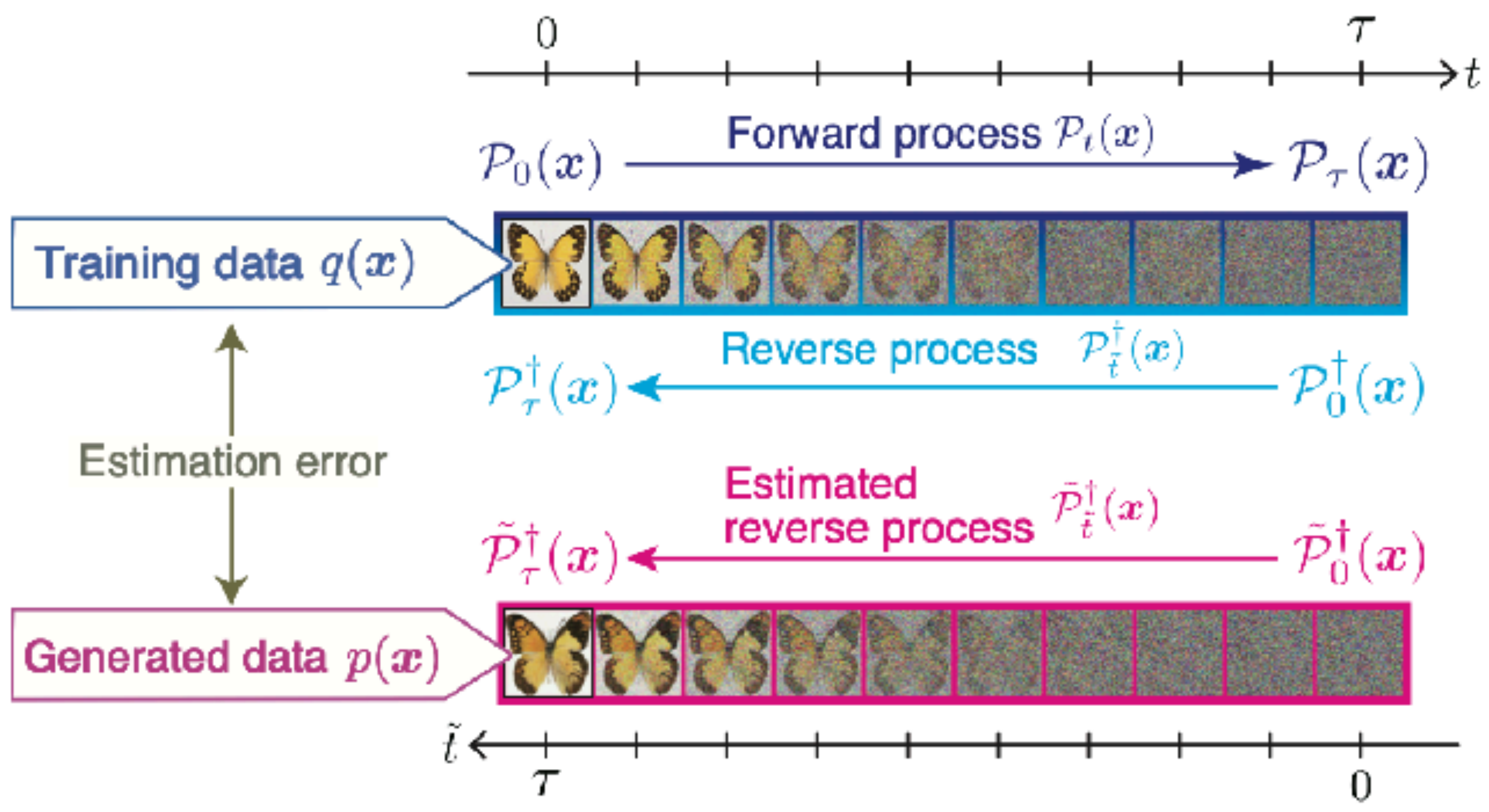


Thermodynamically optimal information erasure in finite time

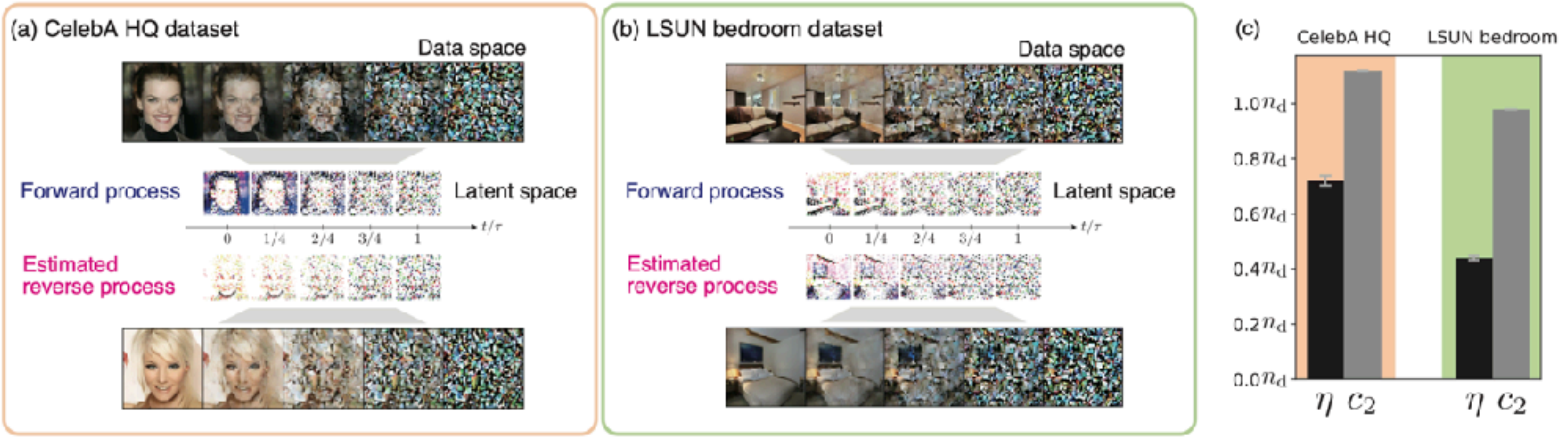
Our works: Possible approach for computer science II

K. Ikeda, T. Uda, D. Okanohara & S. Ito, *arXiv:2407.04495,* to appear in Physical Review X (2025).





Thermodynamically optimal learning method for generative diffusion models.



Summary

- I explained several trade-offs in stochastic thermodynamics. These trade-offs are based on the mathematical properties of the entropy production rate.
- We may quantitatively discuss trade-offs between thermodynamic cost and performance, even in computer science.
- These trade-offs are related to quantities in computer science, such as Kullback-Leibler divergence, Fisher information and p-Wasserstein distance …etc.

Thank you for listening!